—Chapter 9—

Maxwell's Equations

9-1 The Displacement Current

A. SOMETHING IS MISSING

(1) Electric charge in motion is electric current. Because charge is never created or destroyed, the charge density ρ and the current density $\vec{J}$ always satisfy the condition

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \cdots\cdots \text{ equation of continuity}$$

If ρ is constant in time, we have

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0$$

According to Gauss's law,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E}$$

we have

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\left(\epsilon_0 \nabla \cdot \vec{E}\right) = -\epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} = 0$$

Thus, the electric field $\vec{E}$ is constant in time. The current driven by this electric field is called the steady current.

(2) If ρ and $\vec{B}$ change in time, we have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$$

According to Ampère's law,

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0}\left(\nabla \times \vec{B}\right)$$

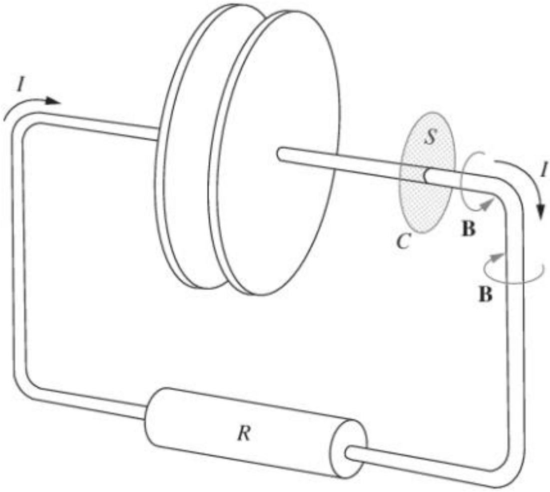
we have

$$\nabla \cdot \vec{J} = \frac{1}{\mu_0} \underbrace{\nabla \cdot \left(\nabla \times \vec{B}\right)}_{=0} = 0$$

We find the contradiction here. Since Ampère's law is only valid for the steady current, there must be a missing term in Ampère's law.

B. THE DISPLACEMENT CURRENT

(1) Consider the line integral of magnetic field around the wire that carries charge away from the capacitor plate

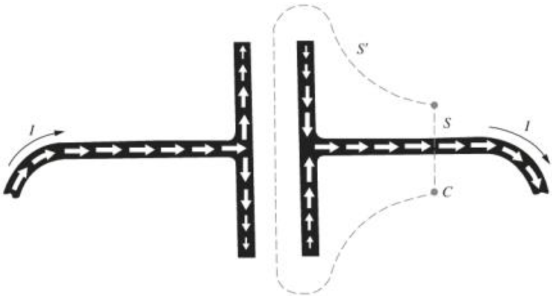


According to Stokes' theorem,

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \int_{\mathcal{S}} \left(\nabla \times \vec{B}\right) \cdot d\vec{a} = \mu_0 I$$

The surface $\mathcal{S}$ passes right through the conductor in which a current I is flowing.

The surface $\mathcal{S}'$ is a surface spanning $\mathcal{C}$,



according to Stokes' theorem, there flows no current through this surface.

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \int_{\mathcal{S}'} \underbrace{\left(\nabla \times \vec{B}\right)}_{\neq 0} \cdot d\vec{a} = \underbrace{\int_{\mathcal{S}'} \mu_0 \vec{J} \cdot d\vec{a}}_{\substack{\text{no current} \\ \text{through} \\ \text{the surface}}} = 0$$

Therefore, on $\mathcal{S}'$, $\nabla \times \vec{B}$ must depend on something other than the current density $\vec{J}$.

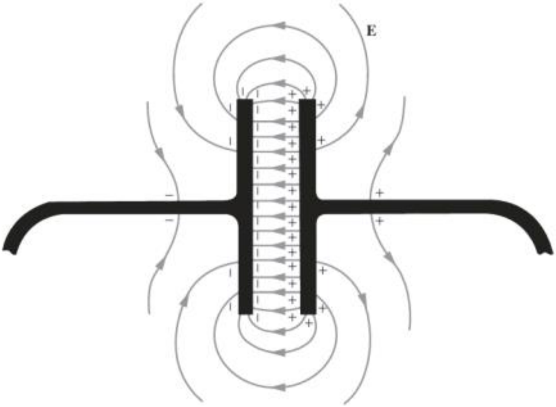
(2) Define a displacement current density $\vec{J}_d$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_d\right) \text{ and } \vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

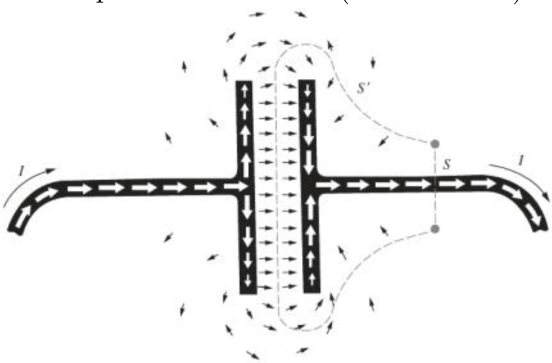
Thus, we obtain

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \int_{\mathcal{S}'} \left(\nabla \times \vec{B}\right) \cdot d\vec{a} = \int_{\mathcal{S}'} \mu_0 \left(\vec{J} + \vec{J}_d\right) \cdot d\vec{a} = \mu_0 \epsilon_o \int_{\mathcal{S}'} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

The electric field at a particular instant.



The displacement-current (black arrows)



9-2 Maxwell's Equations

A. DIFFERENTIAL FORM OF MAXWELL'S EQUATIONS

(1) For fields in the presence of electric charge of density ρ and electric current, that is, charge in motion, of density $\vec{J}$.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \cdots\cdots \text{Gauss's law}$$

$$\nabla \cdot \vec{B} = 0$$

The second expresses the dependence of the magnetic field on the displacement current density, or rate of change of electric field, and on the conduction current density, or rate of motion of charge.
The fourth equation states that there are no sources of magnetic field except currents; that is, there are no magnetic monopoles.

(2) In empty space, the terms with ρ and $\vec{J}$ are zero,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0 \cdots\cdots \text{Gauss's law}$$

$$\nabla \cdot \vec{B} = 0$$

We can write the two induction equations in a symmetric form

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial c\vec{B}}{\partial t}$$

$$\nabla \times c\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

The symmetry between $\vec{E}$ and $c\vec{B}$ is clear.

B. INTEGRATED FORM OF MAXWELL'S EQUATIONS

(1) The integrated form of Maxwell's equations
Use Gauss's divergence theorem and Stokes' theorem,

$$\int_{\mathcal{S}}\left(\nabla\times\vec{E}\right)\cdot d\vec{a}=\oint_{\mathcal{C}}\vec{E}\cdot d\vec{s}\text{ and }\int_{\mathcal{S}}\left(\nabla\times\vec{B}\right)\cdot d\vec{a}=\oint_{\mathcal{C}}\vec{B}\cdot d\vec{s}$$
$$\int_{\mathcal{V}}\left(\nabla\cdot\vec{E}\right)d\tau=\oint_{\mathcal{S}}\vec{E}\cdot d\vec{a}\text{ and }\int_{\mathcal{V}}\left(\nabla\cdot\vec{B}\right)d\tau=\oint_{\mathcal{S}}\vec{B}\cdot d\vec{a}$$

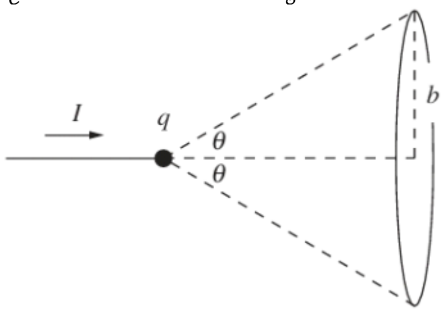
Thus, we obtain

$$\oint_{\mathcal{C}}\vec{E}\cdot d\vec{s}=-\int_{\mathcal{S}}\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}\cdots\cdots\text{ Faraday's law}$$
$$\oint_{\mathcal{C}}\vec{B}\cdot d\vec{s}=\mu_0 I+\mu_0\epsilon_0\int_{\mathcal{S}}\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$$
$$\oint_{\mathcal{S}}\vec{E}\cdot d\vec{a}=\frac{1}{\epsilon_0}\int_{\mathcal{V}}\rho\, d\tau\cdots\cdots\text{ Gauss's law}$$
$$\oint_{\mathcal{S}}\vec{B}\cdot d\vec{a}=0$$

EXAMPLES:

1. A half-infinite wire carries current I from negative infinity $-\infty$ to the origin 0, where it builds up at a point charge with increasing q. Consider the circle, which has radius b and subtends an angle 2θ with respect to the charge. Show that

$$\oint_{\mathcal{C}}\vec{B}\cdot d\vec{s}=\mu_0 I+\mu_0\epsilon_0\int_{\mathcal{S}}\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$$

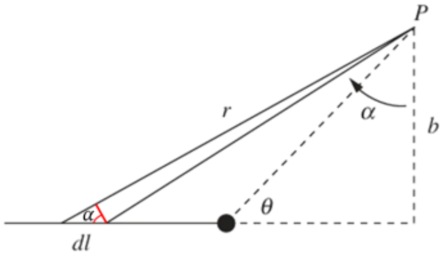


ANSWER:

- L.H.S.: $\oint_{\mathcal{C}}\vec{B}\cdot d\vec{s}$

  Using Biot-Savart law

  $$\vec{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\times\hat{r}}{r^2}$$



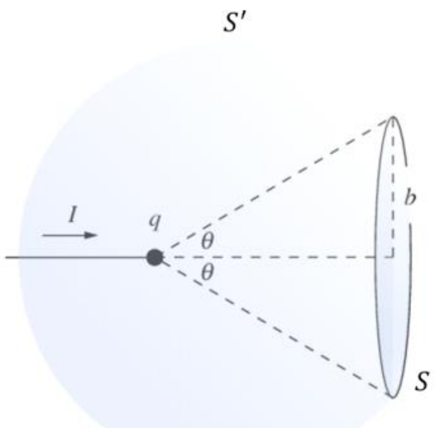
$$b = r\cos\alpha \Rightarrow r = \frac{b}{\cos\alpha}$$
$$dl\cos\alpha = r d\alpha = \frac{b d\alpha}{\cos\alpha} \Rightarrow dl = \frac{b d\alpha}{\cos^2\alpha}$$
$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi}\int \frac{dl\cos\alpha}{r^2} \\ &= \frac{\mu_0 I}{4\pi}\int \frac{\cos^2\alpha}{b^2}\frac{b d\alpha}{\cos^2\alpha}\cos\alpha \\ &= \frac{\mu_0 I}{4\pi b}\int_\alpha^{\pi/2}\cos\alpha\, d\alpha \\ &= \frac{\mu_0 I}{4\pi b}(1-\sin\alpha) \\ &= \frac{\mu_0 I}{4\pi b}(1-\cos\theta) \end{aligned}$$

Therefore, we obtain

$$\oint_{\mathcal{C}} \vec{B}\cdot d\vec{s} = \frac{\mu_0 I}{4\pi b}(1-\cos\theta)2\pi b = \frac{\mu_0 I}{2}(1-\cos\theta)$$

- R.H.S.: $\mu_0 I + \mu_0\epsilon_0 \int_{\mathcal{S}} \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$

  There are two surfaces $\mathcal{S}$ and $\mathcal{S}'$ spanning the closed path $\mathcal{C}$.



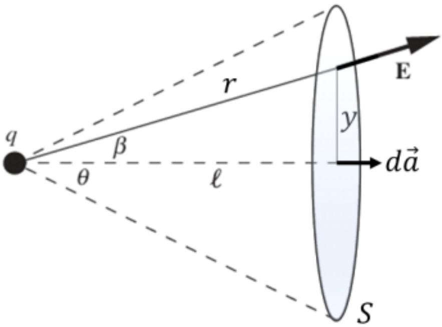
For the surface $\mathcal{S}$, since the current does not pass the surface ($\mathcal{S}$ does not intersect the wire), thus, we only need to calculate

$$\mu_0 \underbrace{I}_{=0} + \mu_0\epsilon_0 \int_{\mathcal{S}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

Since

$$\mu_0\epsilon_0 \int_{\mathcal{S}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0\epsilon_0 \frac{d}{dt} \int_{\mathcal{S}} \vec{E} \cdot d\vec{a}$$

we find the electric field flux through the surface $\mathcal{S}$



he magnitude of the field at the angle β is

$$\ell = r\cos\beta \Rightarrow r = \frac{\ell}{\cos\beta}$$

$$\frac{q}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{\cos\beta}{\ell}\right)^2$$

$$\begin{aligned} da &= 2\pi y dy \\ &= 2\pi r\sin\beta\, d\big(r\sin\beta\big) \\ &= 2\pi \frac{\ell}{\cos\beta}\sin\beta\, d\left(\frac{\ell\sin\beta}{\cos\beta}\right) \\ &= 2\pi\ell^2 \frac{\sin\beta}{\cos\beta}\frac{d\beta}{\cos^2\beta} \end{aligned}$$

The total flux through the surface is

$$\int_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int E\, da \cos\beta$$
$$= \int_0^\theta \frac{q}{4\pi\epsilon_0} \left(\frac{\cos\beta}{\ell}\right)^2 2\pi\ell^2 \frac{\sin\beta}{\cos\beta} \frac{d\beta}{\cos^2\beta} \cos\beta$$
$$= \frac{q}{2\epsilon_0} \int_0^\theta \sin\beta\, d\beta$$
$$= \frac{q}{2\epsilon_0}(1-\cos\theta)$$

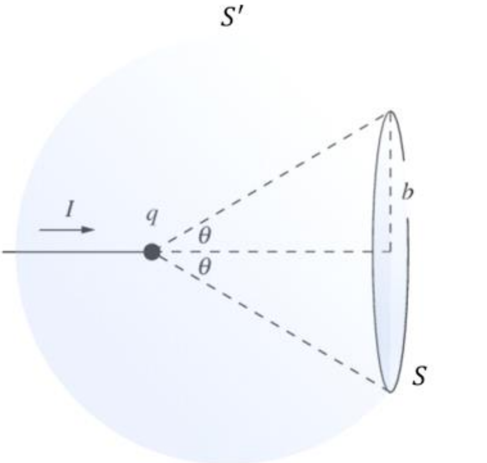
Therefore, the displacement-current is

$$\mu_0\epsilon_0 \int_{\mathcal{S}} \frac{\partial\vec{E}}{\partial t} \cdot d\vec{a} = \mu_0\epsilon_0 \frac{d}{dt}\int_{\mathcal{S}} \vec{E} \cdot d\vec{a}$$
$$= \mu_0\epsilon_0 \frac{1}{2\epsilon_0}(1-\cos\theta)\frac{dq}{dt}$$
$$= \frac{\mu_0 I}{2}(1-\cos\theta)$$

For the surface $\mathcal{S}'$:

The sum of the electric field flux through surfaces $\mathcal{S}$ and $\mathcal{S}'$ equals the total flux emanating from the charge q, which is q/ϵ_0.

$$\oint_{\mathcal{S}+\mathcal{S}'} \vec{E} \cdot d\vec{a} = \int_{\mathcal{S}} \vec{E} \cdot d\vec{a} + \int_{\mathcal{S}'} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$



The flux through the surface $\mathcal{S}'$ is

$$\int_{\mathcal{S}'} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} - \int_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} - \frac{q}{2\epsilon_0}(1-\cos\theta) = \frac{q}{2\epsilon_0}(1+\cos\theta)$$

R.H.S.:

$$\begin{aligned}
\mu_0 I + \mu_0\epsilon_0 \int_{\mathcal{S}'} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} &= \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}\int_{\mathcal{S}'} \vec{E} \cdot d\vec{a} \\
&= \mu_0 I + \mu_0\epsilon_0 \frac{\partial}{\partial t}\frac{q}{2\epsilon_0}(1+\cos\theta) \\
&= \mu_0 I - \frac{\mu_0 I}{2}(1+\cos\theta) \\
&= \frac{\mu_0 I}{2}(1-\cos\theta)
\end{aligned}$$

9-3 Electromagnetic Waves in Vacuum

A. WAVE EQUATION

(1) In empty space, the terms with ρ and $\vec{J}$ are zero, two induction equations are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}_d = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

They constitute a set of coupled first-order differential equations for $\vec{E}$ and $\vec{B}$. Applying curl to both equations give

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \underbrace{\left(\nabla \cdot \vec{E}\right)}_{=0} - \nabla^2 \vec{E} = -\frac{\partial \left(\nabla \times \vec{B}\right)}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \left(\nabla \times \vec{B}\right) = \nabla \underbrace{\left(\nabla \cdot \vec{B}\right)}_{=0} - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial \left(\nabla \times \vec{E}\right)}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Thus, we obtain

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

two decoupled second-order differential equations for $\vec{E}$ and $\vec{B}$. Both equations are called wave equations. So Maxwell's equation imply that empty space supports the propagation of electromagnetic waves, traveling at a speed,

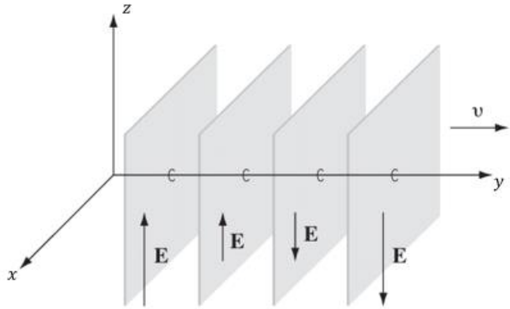
$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3.00 \times 10^8 \text{ m/s} \equiv c$$

which happens to be precisely the velocity of light, c. Thus, we have

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

(2) Suppose that the waves are traveling in the y direction and have no x or z dependence;



these are called *plane waves*, because the fields are uniform over every plane perpendicular to the direction of propagation.
The solutions of electromagnetic wave equations are

$$\vec{E}(y,t) = \vec{E}_0 e^{i(\omega t - k_y \cdot y)}$$
$$\vec{B}(y,t) = \vec{B}_0 e^{i(\omega t - k_y \cdot y)}$$

Since

$\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ in Maxwell's equations,

that is,

$$\nabla \cdot \vec{E}(y,t) = -ikE_{0,y} e^{i(\omega t - k_y \cdot y)} = 0 \Rightarrow E_{0,y} = 0$$
$$\nabla \cdot \vec{B}(y,t) = -ikB_{0,y} e^{i(\omega t - k_y \cdot y)} = 0 \Rightarrow B_{0,y} = 0$$

Thus, electromagnetic waves are transverse: *the electric and magnetic fields are perpendicular to the direction of propagation.*
Since

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

implies a relation between the electric and magnetic amplitudes,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0,x} e^{i(\omega t - k_y \cdot y)} & 0 & E_{0,z} e^{i(\omega t - k_y \cdot y)} \end{vmatrix}$$
$$= -ikE_{0,z} e^{i(\omega t - k_y \cdot y)} \hat{x} + ikE_{0,x} e^{i(\omega t - k_y \cdot y)} \hat{z}$$
$$\frac{\partial \vec{B}}{\partial t} = i\omega B_{0,x} e^{i(\omega t - k_y \cdot y)} \hat{x} + i\omega B_{0,z} e^{i(\omega t - k_y \cdot y)} \hat{z}$$

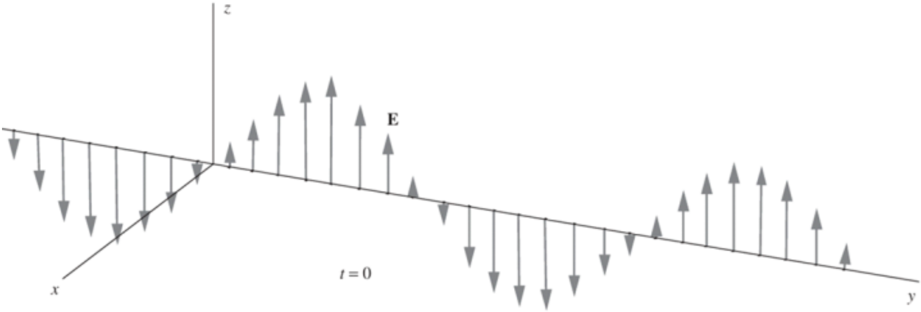
Thus, we obtain

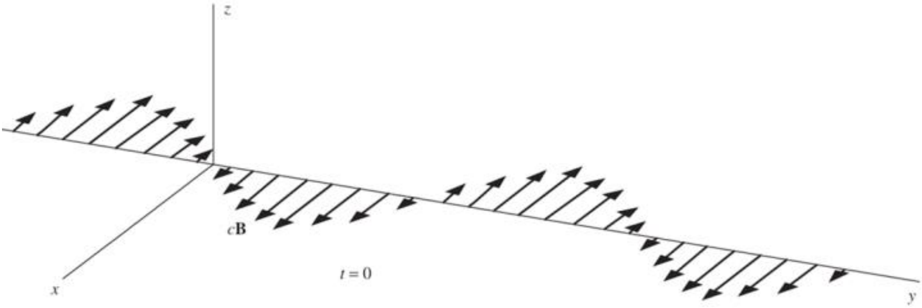
$-k_y \cdot E_{0,z} = \omega B_{0,x}$ and $k_y \cdot E_{0,x} = \omega B_{0,z}$

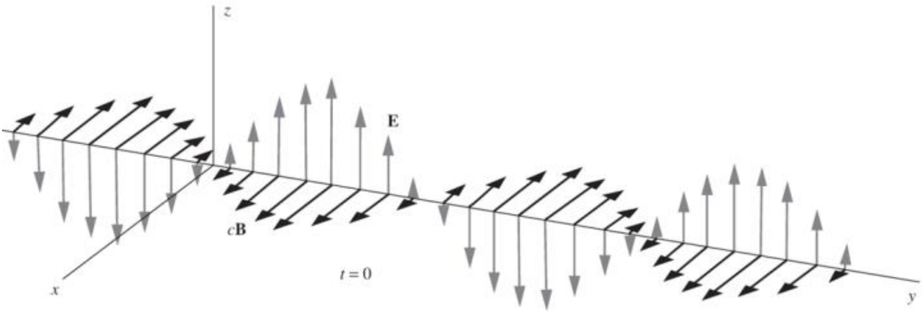
or, more compactly:

$$\vec{B}_0 = \frac{k_y}{\omega}\hat{y} \times \vec{E}_0 = \frac{k}{\omega}\hat{k} \times \vec{E}_0 = \frac{1}{c}\hat{k} \times \vec{E}_0 \text{ where } k = \frac{\omega}{c}$$

Evidently, $\vec{E}$ and $\vec{B}$ are mutually perpendicular.







(3) The monochromatic plane waves traveling in an arbitrary direction.

$$\vec{E}(r,t) = E_0 e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}\hat{n}$$

$$\vec{B}(r,t) = E_0 e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}\frac{\hat{k}\times\hat{n}}{c} = \frac{1}{c}\hat{k}\times\vec{E}$$

where $\hat{n}$ is called the polarization vector.

Since

$$\hat{n}\cdot\hat{k} = 0 \Rightarrow \vec{E}\cdot\hat{k} = 0 \text{ and } \vec{B}\cdot\hat{k} = \frac{1}{c}\left(\hat{k}\times\vec{E}\right)\cdot\hat{k} = 0$$

Both $\vec{E}$ and $\vec{B}$ are transverse.

B. ELECTROMAGNETIC ENERGY AND POYNTING VECTOR

(1) The electrostatic energy is equivalent to a capacitor C with potential difference V between opposite charged plates [c.f.4-4]

$$U_e = \frac{\epsilon_0}{2}\int E^2\, d\tau = \frac{CV^2}{2}$$

The magnetic energy is equivalent to a circuit with self-inductance L containing current I [c.f.7-4]

$$U_m = \frac{1}{2\mu_0}\int B^2\, d\tau = \frac{LI^2}{2}$$

The total energy stored in electromagnetic fields is

$$U = U_e + U_m = \frac{\epsilon_0}{2}\int E^2\, d\tau + \frac{1}{2\mu_0}\int B^2\, d\tau = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$$

(2) Consider some distribution of charges and currents. In small time dt a charge will move $\vec{v}dt$ and, according to the Lorentz force law, the work done on a charge will be

$$dU = \vec{F}\cdot d\vec{l} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\vec{v}dt = q\vec{E}\cdot\vec{v}dt$$

In terms of charge density ρ and current density $\vec{J} = \rho\vec{v}$, we have

$$dU = \rho d\tau\vec{E}\cdot\vec{v}dt = \vec{E}\cdot\rho\vec{v}d\tau dt = \vec{E}\cdot\vec{J}d\tau dt$$

$$\frac{dU}{dtd\tau} = \frac{\partial u}{\partial t} = \vec{E}\cdot\vec{J}$$

The power delivered to the system is

$$\frac{dU}{dt} = \int\frac{\partial u}{\partial t}d\tau = \int\left(\vec{E}\cdot\vec{J}\right)d\tau$$

Thus, $\vec{E}\cdot\vec{J}$ is the power delivered per unit volume.

(3) Express $\vec{E}\cdot\vec{J}$ in terms of the fields,

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot\left(\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

Since

$$\nabla\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\nabla\times\vec{E}\right) - \vec{E}\cdot\left(\nabla\times\vec{B}\right)$$

we obtain

$$\vec{E}\cdot\vec{J}=\frac{1}{\mu_0}\left(\vec{B}\cdot\left(\nabla\times\vec{E}\right)-\nabla\cdot\left(\vec{E}\times\vec{B}\right)\right)-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

Using Faraday's law, we obtain

$$\begin{aligned}\vec{E}\cdot\vec{J}&=\frac{1}{\mu_0}\left(\vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right)-\nabla\cdot\left(\vec{E}\times\vec{B}\right)\right)-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\\&=-\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}-\frac{1}{\mu_0}\nabla\cdot\left(\vec{E}\times\vec{B}\right)-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\\&=-\frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t}-\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}-\frac{1}{\mu_0}\nabla\cdot\left(\vec{E}\times\vec{B}\right)\\&=-\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2\right)-\frac{1}{\mu_0}\nabla\cdot\left(\vec{E}\times\vec{B}\right)\end{aligned}$$

Finally, we can integrate over the volume containing the current and charge,

$$\int\left(\vec{E}\cdot\vec{J}\right)d\tau=-\frac{\partial}{\partial t}\int\left(\frac{\epsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2\right)d\tau-\frac{1}{\mu_0}\int\nabla\cdot\left(\vec{E}\times\vec{B}\right)d\tau$$

Using Gauss's divergence theorem,

$$\int_{\mathcal{V}}\nabla\cdot\left(\vec{E}\times\vec{B}\right)d\tau=\oint_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}$$

we obtain

$$\int\left(\vec{E}\cdot\vec{J}\right)d\tau=-\frac{\partial}{\partial t}\int\left(\frac{\epsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2\right)d\tau-\frac{1}{\mu_0}\oint_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}$$

This is Poynting theorem:

energy lost by fields = energy gained by charges + energy flow out of volume.

(4) Hence we can identify the vector

$$\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

as the energy flux density and it is known as the Poynting vector.

EXAMPLES:

1. In a region of empty space, the power delivered to the system is
$$\frac{dU}{dt}=\int\frac{\partial u}{\partial t}d\tau=\int_{\mathcal{V}}\left(\vec{E}\cdot\vec{J}\right)d\tau=-\frac{1}{\mu_0}\oint_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}=-\oint_{\mathcal{S}}\vec{S}\cdot d\vec{a}$$
Since

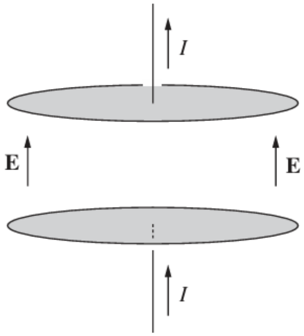
$$\int \frac{\partial u}{\partial t} d\tau = -\oint_{\mathcal{S}} \vec{S} \cdot d\vec{a} = -\int_{\mathcal{V}} \left(\nabla \cdot \vec{S}\right) d\tau$$

hence, we have

$$\frac{\partial u}{\partial t} = -\nabla \cdot \vec{S}$$

This expresses the local conservation of electromagnetic energy.

2. A capacitor has circular plates with radius R and is charged by a constant current I. Find the Poynting vector at radius r inside the capacitor.



ANSWER:

In a region of empty space, we have

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Using Stokes' theorem , we obtain

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \int_{\mathcal{S}} \left(\nabla \times \vec{B}\right) \cdot d\vec{a} = \mu_0 \epsilon_0 \int_{\mathcal{S}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \left(\pi r^2\right)$$

Since

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = B \cdot 2\pi r$$

we obtain the magnetic field as

$$B \cdot 2\pi r = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \left(\pi r^2\right) \Rightarrow \vec{B} = \frac{\mu_0 \epsilon_0}{2} \frac{\partial E}{\partial t} r \hat{\phi}$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \left(\vec{E} \times \frac{\mu_0 \epsilon_0}{2} \frac{\partial E}{\partial t} r \hat{\phi}\right) = -\frac{\epsilon_0}{2} E \frac{\partial E}{\partial t} r \hat{r} = -\frac{\epsilon_0}{4} \frac{\partial E^2}{\partial t} \vec{r}$$

The total power flowing into the cylinder of radius r is then

$$P = -\oint_{\mathcal{S}} \vec{S} \cdot d\vec{a} = \frac{\epsilon_0}{4} \frac{\partial E^2}{\partial t} r (2\pi r h) = \frac{\epsilon_0}{2} \frac{\partial E^2}{\partial t} \left(\pi r^2 h\right) = \frac{dU}{dt}$$

C. ELECTROMAGNETIC WAVE IN DIFFERENT INERTIAL FRAME

(1) According to Faraday's law, a changing magnetic field is accompanied by an electric field, we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is a local relation connecting the electric and magnetic fields in empty space. Remember that the Lorentz-transformation of the electromagnetic fields [c.f.5-5] are

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma\left(\vec{E} + \frac{\vec{v}}{c} \times c\vec{B}\right)_{\perp}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \qquad c\vec{B}'_{\perp} = \gamma\left(c\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)_{\perp}$$

If symmetry with respect to $\vec{E}$ and $c\vec{B}$ is to prevail, we must expect that a changing electric field can give rise to a magnetic field. Thus, we shall have

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial\left(c\vec{B}\right)}{\partial t}$$

$$\nabla \times \left(c\vec{B}\right) = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

This provides that Ampère's law has a missing term and should be modified as

$$\nabla \times \vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

Take the divergence of both sides:

$$\nabla \cdot \left(\nabla \times \vec{B}\right) = \mu_0\nabla \cdot \vec{J} + \mu_0\epsilon_0\nabla \cdot \frac{\partial \vec{E}}{\partial t} = \mu_0\nabla \cdot \vec{J} - \mu_0\nabla \cdot \vec{J} = 0$$

(2) For an electromagnetic wave, $\vec{E}$ and $c\vec{B}$ are perpendicular, i.e.,

$$\vec{E} \cdot \vec{B} = 0$$

and equal in magnitude, i.e.,

$$E^2 - c^2B^2 = 0$$

These two quantities should be invariant in all frames. Thus, we claim that a light wave looks like a light wave in any inertial frame of reference.

(3) Lorentz invariance of $\vec{E}\cdot\vec{B}$

Since

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)_{\perp}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}\right)_{\perp}$$

we have

$$\begin{aligned}\vec{E}'\cdot\vec{B}' &= \vec{E}'_{\parallel}\cdot\vec{B}'_{\parallel} + \vec{E}'_{\perp}\cdot\vec{B}'_{\perp}\\ &= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_{\perp}\cdot\gamma\left(\vec{B}-\frac{\vec{v}}{c^2}\times\vec{E}\right)_{\perp}\\ &= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma^2\left(\vec{E}_{\perp}\cdot\vec{B}_{\perp} - \left(\vec{v}\times\vec{B}_{\perp}\right)\cdot\left(\frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right)\right.\\ &\qquad\left. - \underbrace{\vec{E}_{\perp}\cdot\frac{\vec{v}}{c^2}\times\vec{E}_{\perp}}_{=0} + \underbrace{\left(\vec{v}\times\vec{B}_{\perp}\right)\cdot\vec{B}_{\perp}}_{=0}\right)\end{aligned}$$

Since

$$\begin{aligned}\left(\vec{v}\times\vec{B}_{\perp}\right)\cdot\left(\vec{v}\times\vec{E}_{\perp}\right) &= \vec{E}_{\perp}\cdot\left(\vec{v}\times\vec{B}_{\perp}\times\vec{v}\right)\\ &= \vec{E}_{\perp}\cdot\left(\vec{B}_{\perp}(\vec{v}\cdot\vec{v}) - \vec{v}\left(\vec{v}\cdot\vec{B}_{\perp}\right)\right)\\ &= v^2\vec{E}_{\perp}\cdot\vec{B}_{\perp}\end{aligned}$$

we obtain

$$\begin{aligned}\vec{E}'\cdot\vec{B}' &= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma^2\left(\vec{E}_{\perp}\cdot\vec{B}_{\perp} - \frac{v^2}{c^2}\vec{E}_{\perp}\cdot\vec{B}_{\perp}\right)\\ &= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{1}{1-\frac{v^2}{c^2}}\left(\vec{E}_{\perp}\cdot\vec{B}_{\perp} - \frac{v^2}{c^2}\vec{E}_{\perp}\cdot\vec{B}_{\perp}\right)\\ &= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \vec{E}_{\perp}\cdot\vec{B}_{\perp}\\ &= \vec{E}\cdot\vec{B}\end{aligned}$$

(4) Lorentz invariance of $E^2 - c^2B^2$

$$\begin{aligned}E'^2 - c^2B'^2 &= E'^2_{\parallel} + E'^2_{\perp} - c^2B'^2_{\parallel} - c^2B'^2_{\perp}\\ &= E^2_{\parallel} - c^2B^2_{\parallel} + \gamma^2\left(\vec{E}+\vec{v}\times\vec{B}\right)^2_{\perp} - c^2\gamma^2\left(\vec{B}-\frac{\vec{v}}{c^2}\times\vec{E}\right)^2_{\perp}\end{aligned}$$

Since

$$\left(\vec{E}+\vec{v}\times\vec{B}\right)_\perp^2 = E_\perp^2 + \left(\vec{v}\times\vec{B}_\perp\right)\cdot\left(\vec{v}\times\vec{B}_\perp\right) + 2\vec{E}_\perp\cdot\left(\vec{v}\times\vec{B}_\perp\right)$$
$$= E_\perp^2 + v^2 B_\perp^2$$
$$\left(\vec{B}-\vec{v}\times\vec{E}\right)_\perp^2 = B_\perp^2 + \left(\vec{v}\times\vec{E}_\perp\right)\cdot\left(\vec{v}\times\vec{E}_\perp\right) + 2\vec{B}_\perp\cdot\left(\vec{v}\times\vec{E}_\perp\right)$$
$$= B_\perp^2 + v^2 E_\perp^2$$

thus, we obtain

$$E'^2 - c^2B'^2 = E_\parallel^2 - c^2B_\parallel^2 + \gamma^2\left(E_\perp^2 + v^2B_\perp^2 - c^2\left(B_\perp^2 + \frac{v^2}{c^4}E_\perp^2\right)\right)$$
$$= E_\parallel^2 - c^2B_\parallel^2 + \gamma^2\left(E_\perp^2 - \frac{v^2}{c^2}E_\perp^2\right) - \gamma^2\left(v^2B_\perp^2 - c^2B_\perp^2\right)$$
$$= E_\parallel^2 - c^2B_\parallel^2 + \frac{\left(E_\perp^2 - \frac{v^2}{c^2}E_\perp^2\right)}{1-\frac{v^2}{c^2}} - \frac{c^2}{c^2-v^2}\left(v^2B_\perp^2 - c^2B_\perp^2\right)$$
$$= E_\parallel^2 - c^2B_\parallel^2 + E_\perp^2 - c^2B_\perp^2$$
$$= E^2 - c^2B^2$$